Remember: last time,

I said you could not distribute sums over divergent series.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Using partial fractions,

we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$\neq \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}$$

Since both of these series diverge, but the original converges (partial sums).

The Ratio / Root Tests

(section 11.6)

The two most important things in this class are the ratio test and integration by parts

Recall: shift in attention

We will only ask for convergence or divergence of a series. We won't ask to find the sum.

Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

A series $\sum_{n=1}^{\infty} a_n$ Converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges

Example 1: Let $a_n = \left(-\frac{1}{2}\right)^n$.

Does $\displaystyle\sum_{n=1}^{\infty} \left| \left(-\frac{1}{2}\right)^n \right|$ converge?

$$\sum_{n=1}^{\infty} \left| \left(-\frac{1}{2}\right)^n \right| = \sum_{n=1}^{\infty} \left| (-1)^n \left(\frac{1}{2}\right)^n \right|$$

$$= \sum_{n=1}^{\infty} \left| (-1)^n \right| \left| \left(\frac{1}{2}\right)^n \right|$$

$|(-1)^n| = 1$ for all n, so

$$\sum_{n=1}^{\infty} \left| \left(-\frac{1}{2}\right)^n \right| = \sum_{n=1}^{\infty} \left| \left(\frac{1}{2}\right)^n \right|$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

geometric series, converges $\left(\frac{1}{2} < 1\right)$. This shows $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$ converges absolutely.

Example 2: Let $a_n = \frac{(-1)^n}{n}$.

Then $|a_n| = \frac{1}{n}$, so

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$$ diverges

(p-rule, $p = 1 \leq 1$).

However, we will show

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ converges

(but not today!)

This shows

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges}$$

conditionally.

Note: Any absolutely convergent series is automatically convergent.

Know: what absolute and conditional convergence mean.

The Ratio Test

Let $(a_n)_{n=1}^{\infty}$ be a sequence,

suppose $a_n \neq 0$ for all n.

Then

1) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$,

then $\sum_{n=1}^{\infty} a_n$ converges

absolutely.

2) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$,

then $\sum_{n=1}^{\infty} a_n$ diverges.

3) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$,

then the test fails and you know nothing! (about convergence of series)

Observe similarity between the ratio test and a geometric series

- you prove the ratio test using geometric series!

Example 3:

$$\sum_{n=1}^{\infty} \frac{n+5}{(-e)^n}$$

$$a_n = \frac{n+5}{(-e)^n}$$

$$a_{n+1} = \frac{n+6}{(-e)^{n+1}} = \frac{1}{a_n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{n+6}{n+5} \cdot \frac{(-e)^n}{(-e)^{n+1}} \right| = a_{n+1}$$

(group similar terms together)

$$= \left| \frac{n+6}{n+5} \frac{1}{-e} \right|$$

Since $\dfrac{(-e)^n}{(-e)^{n+1}} = (-e)^{n-(n+1)}$

$$= (-e)^{\cancel{n}-\cancel{n}-1}$$

$$= (-e)^{-1} = \frac{1}{-e}$$

So,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+6}{n+5} \cdot \frac{1}{-e}\right|$$

$$= \left|\frac{n+6}{n+5}\right| \cdot \left|\frac{1}{-e}\right|$$

$$= \frac{n+6}{n+5} \cdot \frac{1}{e}$$

$(n \geq 1)$

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n+6}{n+5}\cdot\frac{1}{e}\right)$$

$$= \frac{1}{e}\lim_{n\to\infty}\frac{n+6}{n+5}$$

$$= \frac{1}{e}\cdot 1$$

$$= \boxed{\frac{1}{e} < 1}$$

So: $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{(-e)^n}$

converges absolutely.

Remark: The ratio test is especially good at handling factorials!

Example 4:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+3)!}$$

$$a_n = \frac{1}{(2n+3)!}$$

$(2n+2+3)!$

$$a_{n+1} = \frac{1}{(2(n+1)+3)!} = \frac{1}{(2n+5)!}$$

$a_n > 0$ for all n, so

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{a_{n+1}}{a_n}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(2n+5)!} \cdot (2n+3)!$$

$$= \frac{(2n+3)!}{(2n+5)!}$$

$$= \frac{\cancel{(2n+3)!}}{(2n+5)\cdot(2n+4)\cdot\cancel{(2n+3)!}}$$

$$= \frac{1}{(2n+5)(2n+4)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(2n+5)(2n+4)}$$

$$= \boxed{0 < 1}$$

This says

$$\sum_{n=0}^{\infty} \frac{1}{(2n+3)!}$$ converges absolutely.

Example 5: $\lim_{n\to\infty} \frac{(23)^n}{n!}$

Use the ratio test (?),

Isn't the ratio test for series,

not sequences?

$a_n = \frac{(23)^n}{n!}$, $a_{n+1} = \frac{(23)^{n+1}}{(n+1)!}$

Again, $a_n > 0$, so

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{a_{n+1}}{a_n}$$

$$\frac{a_{n+1}}{a_n} = \frac{23^{n+1}}{(n+1)!} \cdot \frac{n!}{23^n}$$

$$= \frac{23^{n+1}}{23^n} \cdot \frac{n!}{(n+1)!}$$

$$= 23 \cdot \frac{\cancel{n!}}{(n+1)\cdot(\cancel{n!})}$$

$$= \frac{23}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{23}{n+1} = \boxed{0 < 1}$$

This says

$$\sum_{n=1}^{\infty} \frac{(23)^n}{n!} \text{ converges}$$

absolutely, which says

$$\sum_{n=1}^{\infty} \frac{(23)^n}{n!} \text{ converges!}$$

Since the series converges, its terms must go to zero,

so $$\boxed{\lim_{n\to\infty} \frac{(23)^n}{n!} = 0}$$

Ratio Test $\rightsquigarrow$ sequential limit

$\rightsquigarrow$ l'Hopital's Rule

If you don't know l'Hopital's rule, things could get hard.

Example 6: $\displaystyle\sum_{n=8}^{\infty} \frac{(n+4)!}{(n+6)!}$

There are factorials!

Use ratio test!

$$a_n = \frac{(n+4)!}{(n+6)!} \qquad a_{n+1} = \frac{(n+5)!}{(n+7)!}$$

$a_n > 0$, so we compute

$$\frac{a_{n+1}}{a_n} = \frac{(n+5)!}{(n+7)!} \cdot \frac{(n+6)!}{(n+4)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+5)!}{(n+7)!} \cdot \frac{(n+6)!}{(n+4)!}$$

$$= \frac{(n+5)!}{(n+4)!} \cdot \frac{(n+6)!}{(n+7)!}$$

$$= \frac{(n+5)\cdot(\cancel{(n+4)!})}{\cancel{(n+4)!}} \cdot \frac{\cancel{(n+6)!}}{(n+7)(\cancel{(n+6)!})}$$

$$= \frac{n+5}{n+7}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+5}{n+7} = 1,$$

so test fails :(Why?

The test fails because there are really no factorials.

$$\sum_{n=8}^{\infty} \frac{(n+4)!}{(n+6)!} = \sum_{n=8}^{\infty} \frac{\cancel{(n+4)!}}{(n+6)(n+5)(\cancel{(n+4)!})}$$

$$= \sum_{n=8}^{\infty} \frac{1}{(n+6)(n+5)}$$

which converges using partial fractions & computing partial sums.

$$\frac{1}{(n+6)(n+5)} = \frac{A}{n+6} + \frac{B}{n+5}$$

$$1 = A(n+5) + B(n+6)$$

$n = -5$, $\quad n = -6$

$B = 1 \quad A = -1$

$$\sum_{n=8}^{\infty} \frac{1}{(n+6)(n+5)} = \sum_{n=8}^{\infty} \left(-\frac{1}{n+6} + \frac{1}{n+5} \right)$$

write out terms, start at $n=8$

$$S_1 = -\frac{1}{14} + \frac{1}{13}$$

$$S_2 = \left(-\cancel{\frac{1}{14}} + \frac{1}{13}\right) + \left(-\frac{1}{15} + \cancel{\frac{1}{14}}\right)$$

$$S_3 = \left(\frac{1}{13} - \cancel{\frac{1}{15}}\right) + \left(-\frac{1}{16} + \cancel{\frac{1}{15}}\right)$$

$$S_k = \frac{1}{13} - \frac{1}{k+12}$$

$\lim_{k \to \infty} S_k = \frac{1}{13}$, this is the sum of the series.

The Root Test

Let $(a_n)_{n=1}^{\infty}$ be any sequence

If

1) $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$,

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

2) $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L > 1$,

then $\sum_{n=1}^{\infty} a_n$ diverges.

3) If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$,

the test fails and

you know nothing

(about the series).

When to use the root test?

When you see terms to

the n^{th} power, and

only then!

Example 7: $\displaystyle\sum_{n=3}^{\infty} \left(\frac{\ln(n)}{n^2} \right)^n$

Again, $a_n > 0$, so

$$|a_n|^{\frac{1}{n}} = (a_n)^{\frac{1}{n}} = \left(\left(\frac{\ln(n)}{n^2} \right)^{\cancel{n}} \right)^{\cancel{1/n}}$$

$$= \frac{\ln(n)}{n^2}$$

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln(n)}{n^2}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = \lim_{n\to\infty} \frac{1}{2n^2}$$

$$= \boxed{0 < 1}$$

This says the series converges absolutely.